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1989 J. Phys. A: Math. Gen. 22 3991

(http://iopscience.iop.org/0305-4470/22/18/032)

J. Phys. A: Math. Gen. 22 (1989) 3991-4005. Printed in the UK

Block spin transformations in the operator formulation of two-dimensional Potts models

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Received 18 August 1988, in final form 22 March 1989

Abstract. We show that statistical mechanical models, such as the q-state Potts model, whose transfer matrix may be written in terms of the Hecke algebra $H_n(q)$, respond to block spin scale changes at a certain critical point in a way which may be interpreted as an algebra homomorphism. Part of the structure of the algebra, and hence part of the transfer matrix spectrum of the model, is preserved provided $q = 4\cos^2(\pi/r)$ (r integer). These partial realisations of scale invariance are of a distinct type for each r, characterised in some cases by an endomorphism of the algebra. Such an endomorphism would generate a ring of integer dilatations in the conformal symmetry of the field theory limit. We discuss the realisation of other aspects of conformal symmetry in the lattice algebra.

The homomorphisms we describe preserve the Yang-Baxter equations for all r and all temperatures.

1. Introduction

It has been pointed out by several authors that the critical point field theory limit of various two-dimensional statistical mechanical models labelled by a parameter $q = 4\cos^2(\pi/r)$ (r integer) is given by the conformal field theory (CFT) with central charge

$$c = 1 - 6/[r(r-1)] \tag{1}$$

(for a recent list of references see Cardy (1987)). Specifically, these models include the appropriate q-state Potts models, six-vertex models and the Andrews-Baxter-Forrester (1984) models.

Evidence for this identification consists mainly in the charged Coulomb gas (CG) picture of such models (see Nienhuis 1987). The appropriate Coulomb gases have been associated with CFT by Dotsenko and Fateev (1984). The Coulomb gas picture is not in any doubt, because of the quantity of supporting evidence derived from independent sources. However, it relies on various universality assumptions. These are borne out for integer r, but one has to work hard to make the CG renormalisation group analysis differentiate between integer and (say) irrational r (cf Belavin *et al* (BPZ) 1984).

In contrast, the above-mentioned models may be constructed exactly in terms of representations of quotients of the A_n-type Hecke algebras, $H_n(q)$ (Hoefsmit 1974), each of which has the property that it has a generic structure for irrational r and a rich sequence of special structures for rational r (among which the integers play a further special role—see Martin (1988)). This is strikingly analogous to the observed behaviour of the Virasoro algebras labelled by c (Friedan *et al* 1984). It seems worthwhile, therefore, to try and recouch the renormalisation group arguments in this framework. Furthermore, unlike the conformal structure, the algebraic formulation we will use is preserved in significant part away from criticality and for all q. It therefore provides a much more general tool with which to examine the response of statistical systems to scale changes.

In the present paper we discuss the response of these lattice models, in their general operator algebraic construction, to dilatations generated by block spin renormalisation. Perhaps the most topical aspect of the response to scale transformations in general is, however, the possible instance of scale transformation invariance. We therefore focus most attention on the conditions for manifestation of dilatation invariance in the algebraic construction, specifically of the critical models.

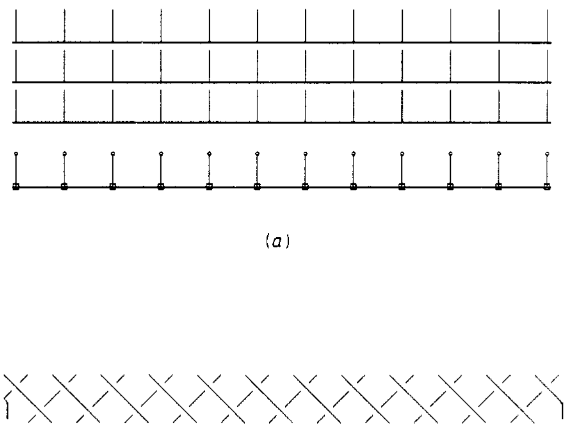
Since each appropriate Hecke algebra quotient builds a model with a conformal field theory limit, then ultimately the whole Virasoro algebra should be constructed directly as a subalgebra of that quotient, in each case. So far, however, this construction has been realised only for the unitarisable $c=\frac{1}{2}$ case (see, for example, Martin 1989). On the other hand, global scale changes, translations and rotations form a non-trivial subset of the conformal transformations. Among these, only certain translations are automorphisms of the lattice, and none of them are obvious automorphisms of the algebra we will introduce, so they provide a good testing ground for the algebraic approach. We work with models parametrised initially by arbitrary q and arbitrary temperature. We exhibit the need to restrict to the above special values of q (the 'Beraha' values (Baxter 1982)) and to critical points. Our observations also appear to distinguish the nature of the invariance for distinct values of c in the lattice framework. In the field theory limit the response to global scale changes should not be sensitive to the central charge (see Cardy 1987). However, we never take this limit, so such sensitivity is not ruled out. Local scale changes are discussed more briefly.

Note that the CG has infinite range interactions, and so is not itself amenable to the local transfer matrix treatment (reviewed below) which reveals the algebraic content of the Potts and other such models (see also Schultz *et al* (SML) 1964).

2. Block spin transformations and transfer matrices

In the simplest scenario for block spin renormalisation in equilibrium statistical mechanics we have a partition function written in terms of the classical Hamiltonian for some set of fields (lattice 'spins' resolved at some scale a) with various coupling parameters $\{\beta\}_a$, and the same partition function written in terms of a set of fields resolved at ba ($b\neq 1$) with couplings $\{\beta(\{\beta\}_a)\}_{ba}$. The fixed points of the implied coupling transformations (renormalisations) are associated with scale-invariant critical points, and the transformations linearised at these points give critical exponents (see, for example, Ravndal 1976, Swendsen 1984).

Consider a regular two-dimensional many-body system resolved at some given length scale in terms of spins on a lattice. The transfer matrix T for such a system is defined as follows. The matrix element $(T)_{ij}$ is the exponentiated classical Hamiltonian for a single one-dimensional layer of the lattice system with the spin configurations on the 'leading' and 'trailing' edges or faces of the layer determined by i and j, respectively (see, for example, Kogut 1979). Figure 1(a) illustrates the example of the Potts model. We will refer to the spins associated with the leading edge as 'outgoing' and the trailing edge as incoming. The boundary conditions at the ends of the layer



(a)

(b)

Figure 1. (*a*) Adding a one-dimensional layer to the two-dimensional lattice in the Potts model case. Sites marked ○ are associated with q-state spins in the trailing edge; sites marked □ are in the leading edge. (*b*) The braid corresponding to $T(-e^{\theta}, -e^{-\theta})$ for an $N = 12$ site lattice.

need not concern us for the moment. In the transfer matrix formalism the statistical mechanical partition function for a two-dimensional model on a lattice composed of $M = mb$ (with m, b integers) layers, with overall leading and trailing edge boundary conditions given by i and j, is thus written

$$Z = \langle i|T^M|j\rangle = (T^M)_{ij}. \tag{2}$$

The brackets identify the boundary conditions as vectors in the configuration space of a layer edge. Such a vector is often written $\langle i| \in V\otimes V\otimes V\otimes\ldots\otimes V$ where V is the space of configurations of a single spin and the product is over spins in an edge. For example, $Z = \mathrm{Tr}\,(T^M) = \Sigma_i \langle i|T^M|i\rangle$ corresponds to periodic boundary conditions. We will suppress explicit reference to boundary states in what follows, so $Z = \langle T^M\rangle$.

Thus we require, for scale invariance under a scale change b,

$$\langle (T_a(\{\beta\}_a))^M\rangle \sim \langle (T_{ba}(\{\beta\}_{ba}))^{M/b}\rangle \tag{3}$$

or, more strongly,

$$(T_a(\{\beta\}_a))^b \sim (T_{ba}(\{\beta\}_{ba})) \tag{4}$$

where T_a is the transfer matrix for fields resolved at scale a.

In general the set of couplings $\{\beta\}_a$ could include any possible lattice interactions (some may be eliminated by symmetry). However we may try and approximate the transformations by manually turning off some interactions at each stage (see Cardy 1987). One non-trivial approximation of this kind is to consider just two (longitudinal and transverse nearest neighbour) coupling parameters.

For example, the classical q-state Potts model Hamiltonian may be written

$$H = \sum_{\langle ij\rangle} \beta_{ij}\delta_{s_i,s_j}$$

where the spins s_i take values from $\{1, 2, \ldots, q\}$ on the sites i of the square lattice (of figure 1(a)), and the sum is over nearest-neighbour bonds (so $\beta_{ij} = \beta_1$ if i, j are separated by a vertical bond, say, and β_2 if by a horizontal bond). Another example is the six-vertex model, in which the degrees of freedom $s_b \in \{1, -1\}$ reside on the lattice bonds b of a square lattice 'medial' to that of figure 1(a) (Baxter 1982), and the interactions on the sites. The interactions in this case are usually given by defining the Boltzmann weights for the various possible configurations at a site. Referring to two adjacent bonds impinging on a site as incoming, and the other two outgoing, then the weights are set to zero unless the outgoing states are a permutation of the incoming ones. This leaves six non-zero interactions (hence six-vertex model). The interactions are usually taken to be invariant under $s_b \to -s_b$ for all b, so there are three distinct interaction weights to specify per site. One of these may be taken to be unity without loss of generality. The relationship between the other two is determined by q (see Baxter 1982), leaving one variable coupling parameter. The two-parameter version of the model has different triples of weights for sites on the odd and even sublattices.

We define the 'single-interaction transfer matrix' $t_i(x)$ for such models so that $(t_i(x))_{jk}$ is the exponentiated classical Hamiltonian for a single interaction at position i (with coupling parameter x) in a lattice layer with leading and trailing edge configurations given by j and k. It is then possible to write the one-dimensional layer transfer matrices for a large class of two-parameter lattice models, including the above models, in the form

$$T(x_1, x_2) \propto \prod_{j=1}^{N} t_{2j-1}(x_1^{-1}) \prod_{j=1}^{N-1} t_{2j}(x_2) = ST'(x_1, x_2)S^{-1} = S\left(\prod_{j=1}^{2N-1} t_j((x_j)^{(-1)^j})\right)S^{-1}. \tag{5}$$

Here $2N-1$ is the total number of interactions included in the layer (so $N=12$ in figure 1(a)). For the Potts model, the factors t_i each introduce the effect of one 'horizontal' (i even) or vertical (i odd) lattice bond interaction into the Hamiltonian. The matrices T and T' just differ, therefore, in the order in which the effect of various interactions is incorporated into the Hamiltonian, and hence in the effective layering direction (for the Potts model T builds layers parallel to the horizontal bonds, while T' builds staircase layers at 45°). The non-singular S matrix is a (readily calculable) function of x_1 and x_2, and $x_{\text{odd,even}} = x_{1,2}$. These parameters are related to the original Hamiltonian parameters by $x = (\exp\beta - 1)q^{-1/2}$ in the Potts case.

A critical point in the chosen two-parameter submanifold of general coupling parameter space is any point on the line

$$x_1 x_2 = 1 \tag{6}$$

(Baxter 1982). We will write the one-parameter family of critical transfer matrices as $T'(1/x, x) = T'(x)$.

We should make a few general points. Suppose there are s p-state spins in each incoming and outgoing edge of a layer (in the Potts case $s=N$, $p=q$, in six-vertex $s=2N$, $p=2$), and the interaction at position i depends on the configuration of t nearby incoming and t outgoing spins. Then we get the following.

(i) The matrix $t_i(x)$ acts trivially on all the subspaces of configurations of spins not involved in interaction i, that is $t_i(x) = 1\otimes 1\otimes \ldots \otimes 1 \otimes R_i(x) \otimes 1 \ldots \otimes 1$, where the product contains $(s-t)$ p-dimensional unit matrices and R_i is a p^t-dimensional matrix.

(ii) In a general model (with local interactions) $R_i(x)$ could assign a different weight to each possible configuration of the variables involved in i. It would be necessary, in general, to build R_i out of representations of a p^t-dimensional 'local' matrix algebra. We call this local because it covers only a single interaction.

The models we consider are characterised by (iii) the fact that $R_i(x)$ is built out of representations of a two-dimensional subalgebra—the simplest non-trivial special case; and by (iv) the existence of a Yang-Baxter equation (see later).

Specifically, in the models of interest $t_i(x)$ may be written in the form

$$t_i(x) \propto 1 + xU_i$$

where 1 is the unit matrix of appropriate dimension. The $2N-1$ matrices $\{U_i\}_{2N-1}$ obey the Temperley-Lieb algebra relations

$$U_iU_i = \sqrt{q}U_i \tag{7a}$$

$$U_iU_jU_i - U_i = 0 \qquad i-j=\pm 1 \tag{7b}$$

$$[U_i, U_j] = 0 \qquad i-j \neq \pm 1 \tag{7c}$$

(Temperley and Lieb (TL) 1971) in our examples, or more generally those of the Hecke algebra $H_{2N-1}(q)$ (see Hoefsmit 1974) in which (7*b*) is replaced by the weaker but not necessarily inconsistent condition

$$U_iU_{i+1}U_i - U_i = U_{i+1}U_iU_{i+1} - U_{i+1}. \tag{8}$$

This latter relation pertains, for example, in a generalisation of the six-vertex models in which the two-state bond variables are replaced by $p>2$ state variables. In this paper we will assume the TL relations on $\{U_i\}_{2N-1}$ unless otherwise stated. To summarise: the 'local' relation (7*a*) comes from point (iii); relation (7*c*) from point (i), and (7*b*) or (8) from point (iv). We will see later that our special models renormalise in general to models in which R_i is built from a larger local algebra.

Up to degeneracy the algebra defined by the generators and relations above determines the spectrum of T (Martin 1988) and we may think of replacing the matrices with abstract operators. For example, the Temperley-Lieb algebra has a set of primitive idempotents I_i ($i=0,1,2,\ldots,k-1$ where k is the sum of dimensions of distinct irreducible representations) for which a map from any element of the algebra κ to a scalar $R_i(\kappa)$ is defined by $I_i\kappa I_i = R_i(\kappa)I_i$. If κ is the transfer matrix raised to a large power then $R_i(\kappa)$ will be numerically dominated by the largest magnitude eigenvalue whose eigenvector is not orthogonal to I_i. A particularly useful example of this is the idempotent

$$I_0 = \prod_{j \text{ odd}} (U_j/\sqrt{q}).$$

It is easy to see that this is a primitive idempotent on application of the TL relations. In the physical coupling region $R_0(T^M)$ is dominated by the largest eigenvalue of T in the Potts representation, and hence gives the partition function for some reasonable set of boundary conditions (actually $\beta_1 = 0$ at the boundaries). Other eigenvalues in the spectrum of T determine the long-distance correlations of various observables in the system (see Kogut 1979).

Note that the definition of I_0 works for any lattice size N. Similarly, other idempotents may be associated with observables in an N-independent way. This gives a chain of physically motivated correspondences, connecting the irreducible representations responsible for a given observable at each different lattice size (see later).

In general we would expect a block spin transformation to map from the two-parameter model to one containing more complicated couplings. In particular, we would not expect the operator algebraic structure above to be preserved. However, we will be optimistic and consider the possibility of finding a fixed point somewhere in the two-parameter space, in which case it must be on the critical surface given by $x_1 x_2 = 1$ (with x_1 and x_2 regarded as complex variables). Such a point is supposed to be universal (in the sense of Kogut (1979)) with other critical points on the surface. It is known, for example, that the divergence of the Ising model specific heat depends on the approach of the product $x_1 x_2$ to unity, and not on any properties of x_1, x_2 separately (see, for example, Baxter 1982). Specifically we will look for a point where at least part of the algebraic structure of the renormalised model is recognisable from the original model (i.e. a weaker version of (4) in which at least part of the spectrum is the same).

3. Yang–Baxter equations

For the models we refer to above the spectrum of T has not been calculated in general. One property which generally leads to computability of this spectrum ('solvability' of the model) is commutativity of transfer matrices defined at different couplings (see Baxter 1982). On the critical surface this is (up to possible boundary terms) a consequence of the Yang–Baxter (YB) equations:

$$t_i(x)t_{i+1}(y)t_i(z(x, y)) = t_{i+1}(z(x, y))t_i(y)t_{i+1}(x). \tag{9}$$

To see this consider repeated applications of these relations to the product

$$p(x, y, z) = t_1(x)t_2(x)t_3(x) \ldots t_n(x)t_1(y)t_2(y)t_3(y) \ldots t_n(y)(t_n(y))^{-1}t_n(z)t_n(x). \tag{10}$$

If we restrict attention to the critical surface then $p(x, y, z)$ is, up to boundary terms, a product of transfer matrices $T'(x)T'(y)$. Using the YB equations we find $p(x, y, z) = t_1(x)t_1(z)(t_1(y))^{-1}T'(y)T'(x)$.

Applying the TL relations to $t_i = 1 + xU_i$ we find that the YB equations are satisfied with

$$z(x, y) = (y - x)/(1 + x\sqrt{q} + xy)$$

or

$$z(-e^{\theta} - \varepsilon, -e^{\theta} - \delta) = (\varepsilon - \delta)/(-e^{-\theta}\varepsilon + e^{\theta}\delta)$$

where $e^{\theta} + e^{-\theta} = \sqrt{q}$ (it is also useful to define $Q = e^{2\theta}$). The *critical* models have thus been solved (Baxter 1982).

Note the trivial case of the YB equations here, $x = y \neq -e^{\pm\theta}$ implies $z = 0$, and the special case $x = y = z = -e^{\theta}$. Writing $t_i(-e^{\theta}) = t_i$ these special single-bond transfer matrices then satisfy (amongst other things) the relations for generators of the $2N$ string braid group $\mathscr{B}_{2N}$ (Birman 1974). That is, with

$$t_i^{\pm 1} = t_i(-e^{\pm\theta}) = (1 - \exp(\pm\theta)U_i) \tag{11}$$

we have

$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \tag{12a}$$

(see, for example, Temperley 1986). The additional relations for this alternative set of generators of the Hecke algebra are (from (7) and (8))

$$t_i t_{i+j} = t_{i+j} t_i \qquad j > 1 \tag{12b}$$

$$t_i^2 = (1 - e^{2\theta})t_i + e^{2\theta}. \tag{12c}$$

In other words the TL or Hecke relations of (7) and (8) imply solutions of the YB equations, and the YB equations have fixed points which imply realisations of the braid relations. However, realisations of the braid relations do not necessarily imply realisations of the Hecke relations, since the quadratic relation need not be preserved (see later).

One small but non-trivial aspect of conformal symmetry, discrete translation invariance, has been implicitly realised in the above operator formalism. Translation within the transfer layer may be achieved by the conjugation

$$U_i = T'(-e^{\theta}, -e^{-\theta})U_{i-1}(T'(-e^{\theta}, -e^{-\theta}))^{-1}. \qquad (12d)$$

In particular, a $U_0 = U_{2N}$ may be defined within H_{2N-1} by conjugating U_{2N-1}. This, incidentally, gives a generalisation of the formalism to a periodic lattice. Note that the above conjugation is a similarity transformation on any representation, so the algebra automorphism maps representations to themselves. Translation by one lattice spacing is achieved by applying the above conjugation twice (a single conjugation has the effect of a duality transformation). There are other ways of achieving such a translation, each corresponding to a different choice of seam on closing the periodic boundary.

Translations in the layering direction are achieved by conjugating by T itself, i.e. an expectation value on the $N \times M$ site lattice (specifically an unsubtracted correlation of local observables $\mathcal{O}_{ij}$ between some origin (0, 0) and another point (i, j)) is given by

$$\langle \mathcal{O}_{00}\mathcal{O}_{ij}\rangle_M = \langle T^{M/2}\hat{\mathcal{O}}_0 T^j (T')^{2i}\hat{\mathcal{O}}_0(T')^{-2i}T^{-j}T^{M/2}\rangle / Z$$

where $(\hat{\mathcal{O}}_0)_{xy} = O(x)\delta_{x,y}$ gives the result of a measurement $\mathcal{O}$ (at the origin point of the layer) on layer configuration x. Here and subsequently we write T' without arguments for T' taken with the couplings in (12d).

4. Cabling transformations

The transformations above act as translations at any (x_1, x_2) (although T is x dependent). In other words, discrete translation invariance is a symmetry which survives away from criticality. By restricting to the critical point of (12d) we can also realise a block spin transformation in the operator framework. The transfer matrix in this case is represented by the braid shown in figure 1(b) (see also part (i) of the appendix). Taking T^2 (for $b = 2$ in (4)) we have the braid shown in figure 2, and so on.

Now consider $\tilde{T}$ defined by the braid shown in figure 3 (we will call this a cabling of strings in pairs) and compare T^{2m} with $(\tilde{T})^m$ for large m and N. The differences

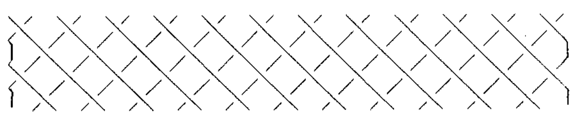


Figure 2. The braid corresponding to T^2.

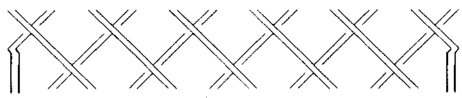


Figure 3. The braid corresponding to $\tilde{T}$.

are purely boundary effects. But in $\tilde{T}$ pairs of strings manifestly preserve their orientation (in other words, we have integrated out on the scale of the corresponding interactions—see (5)). The fixed-point criterion then simply requires the equivalence represented in figure 4 (at least on the infinite lattice), where these diagrams are to be interpreted at the level of the transfer matrix. That is

$$(1-e^{\theta}\tilde{U}_i)=(1-e^{\theta}U_{2i})(1-e^{\theta}U_{2i-1})(1-e^{\theta}U_{2i+1})(1-e^{\theta}U_{2i}) \tag{13a}$$

or, for example,

$$\tilde{U}_1 = U_1+U_3+(1+e^{2\theta})U_2-e^{\theta}(\{U_2, U_1\}+\{U_2, U_3\}+U_1U_3) + e^{2\theta}\{U_2, U_1U_3\}-e^{3\theta}U_2U_1U_3U_2 \tag{13b}$$

where $\{\tilde{U}_i\}$ should be another Temperley-Lieb or Hecke algebra (with roughly half as many generators) *if* dilatation invariance is realised. Equating coefficients we find that the requirement that the new operators satisfy the Hecke relations determines

$$\exp(2\theta)=1 \tag{14}$$

that is, $q=4$. To see this note, for instance, that the coefficient of U_{2i+1} must be the same in $\tilde{U}_i^2$ and $(e^{\theta}+e^{-\theta})\tilde{U}_i$ (relation (7*a*)). The coefficient of U_{2i+1} in $\tilde{U}_i^2$ comes from terms in the expansion of (13) beginning and ending with U_{2i+1}, that is $e^{3\theta}+e^{-\theta}=e^{\theta}+e^{-\theta}$. The same condition is obtained for coefficients of U_{2i-1} (by symmetry) and U_{2i}. Altogether we have (putting $i=1$, for example)

$$(\tilde{U}_1)^2-\sqrt{q}\tilde{U}_1=(e^{3\theta}-e^{\theta})[\tilde{U}_1+(e^{3\theta}-e^{-\theta})U_1U_3-e^{4\theta}\{U_2, U_1U_3\}+U_1U_2U_3+U_3U_2U_1 + (e^{3\theta}+e^{\theta})e^{2\theta}U_2U_1U_3U_2]. \tag{15}$$

We are writing these expressions out explicitly because they will be useful later on. It is easy to see that if relation (7*a*) is satisfied then the relations (8) and (7*c*) follow from the form of equation (13). That is, noting from the diagrams that we automatically have

$$\tilde{t}_i\tilde{t}_{i+1}\tilde{t}_i=\tilde{t}_{i+1}\tilde{t}_i\tilde{t}_{i+1} \tag{16}$$

(see also Birman 1974) we then have relation (8), and so on. Note that *this* observation is independent of the number of strings in the cable, n say, although we have taken $n=2$.

Using the same diagrammatic notation we can look for fixed points under the blocking represented in figure 5 and defining $U^{(3)}$ (in a notation in which $\tilde{U}$ becomes

Figure 4. The equivalence $t_2t_1t_3t_2=\tilde{t}_1$.

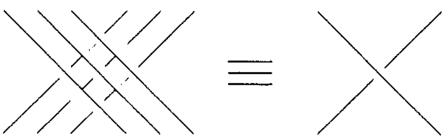


Figure 5. The equivalence $t_3t_2t_4t_1t_3t_5t_2t_4t_3=t_1^{(3)}$.

$U^{(2)}$), and so on. The obvious generalisation of this procedure to n string cables gives (with $x=-e^{\theta}$)

$$t_i^{(n)}(x)=\left(\prod_{k=1}^{n}\prod_{j=-(k-1)}^{(k-1)}{}' \, t_{ni+j}(x)\right)\left(\prod_{k=n-1}^{1}\prod_{j=-(k-1)}^{(k-1)}{}' \, t_{ni+j}(x)\right) \tag{17}$$

(where the order of the product Π_k matters and the index j in the product Π_j' is incremented in steps of 2), and hence $U_i^{(n)}$.

Consider, for a moment, the effect of generalising away from $x=-e^{\theta}$. The $\{t_i(x)\}$ no longer satisfy the braid relations, but satisfy the YB equations. The diagrammatic notation may be generalised to represent these by figure 6. It is then obvious, by repeated application of the Yang-Baxter equations, that models obtained by the replacement $t_i^{(n)}(x)\to t_i(x)$ continue to satisfy the YB equations for all temperatures with the same functional form for $z(x, y)$. This means that our cabling procedure generates many new solvable models (solvable in the sense of computable spectra). This procedure is basically equivalent to the fusion procedure of Date *et al* (1987), but works for any model built using the Hecke algebras. Note that the cabling factor may vary from braid to braid without affecting the generalised YB equation. We can also attach an idempotent for the n-string subalgebra (Markin 1989) to each n-string cable without disturbing the braid relations.

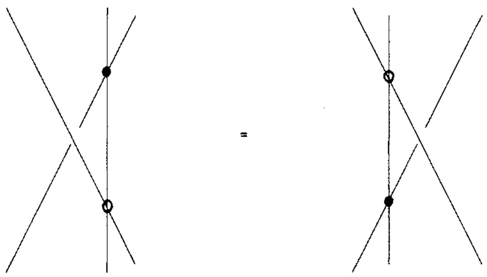


Figure 6. The Yang-Baxter equation.

We have shown that the block spin transformations or cablings (17) map from solvable models to solvable models. What we want now is to find models which are taken to equivalent models by such maps. That is, the fixed points of the maps. These occur when $\{U_i^{(n)}\}$ satisfy the same relations as $\{U_i\}$. Equation (14) gives such a model for $n=2$ (see later).

The weak version of (4) requiring part of the spectrum of the 'equivalent' models to be the same corresponds to allowing the set of relations on the $\{U_i\}$ to be added to, in order to recover the Hecke relations at the $\{U_i^{(n)}\}$ level, without requiring that corresponding relations be added to the set for $\{U_i^{(n)}\}$. For example, if the condition $Q=e^{2\theta}=1$ from (14) is not satisfied then the additional relation:

$$\begin{aligned} t_2t_1t_3t_2t_1t_3=[Q/(1-Q)]\{(1+Q+Q^2)+[(1-Q+Q^2)/Q]t_2t_1t_3t_2-Q^2t_2 \\ -Q(t_2t_1t_2+t_2t_3t_2)-t_1t_2t_3t_2t_1\} \end{aligned} \tag{18}$$

is necessary and sufficient to ensure that $t_i^{(2)}$ (and hence $U_i^{(2)}$) has the correct quadratic relation. As we will see later, the non-trivial solutions to this kind of extra condition restrict q severely in general. The dimension of $H_{2N-1}(q)$ is $(2N)!$. The left-hand side of the above relation is the longest word, in these generators, in $H_3(q)$. After some work we find that, generically (i.e. for $q \neq 4, 0, 1, 2$—see Martin (1989)), this relation implies $t_i = 1$.

Putting $n = 3$ in (17) we find that the coefficients of $U_{3i\pm 2}$ in the right- and left-hand sides of relation (7*a*) for the *new* operators are not equal unless

$$\exp(4\theta) = 1 \qquad \text{or} \qquad q = 4, 0. \tag{19}$$

Generalising to $n \times n$ blocking we find that this necessary condition determines

$$\exp[2(n-1)\theta] = 1. \tag{20}$$

This means that

$$\theta = 2m\pi \mathrm{i}/2(n-1) \tag{21}$$

or

$$q = (\mathrm{e}^{\theta} + \mathrm{e}^{-\theta})^2 = 4\cos^2[m\pi/(n-1)] \qquad \text{with } m \text{ integer.}$$

In other words the possible scale invariant critical points at this stage are indeed those with the Beraha q values.

5. The case $q = 4$

Unfortunately we find, by further direct calculation, that this necessary condition is not generally sufficient. The Hecke relations are only exactly realised for the new operators, without extra conditions at the $\{U_i\}$ level, when $q = 4$. Nonetheless this analysis reveals that, at $q = 4$, at least the thermodynamic limit model has integer dilatation invariance (in the weak sense of (4), with $b = n$, $n \in \mathbb{Z}$). This is enough to imply a massless field theory limit (Cardy 1987), provided that the representation of the algebra responsible for the free energy survives in the spectrum of the renormalised model. To see that it does, note that in the case $q = 4$ we have $t_i^2 = 1$, so the $\{t_i\}$ satisfy relations for the generators of the permutation group on $2N$ objects, S_{2N}.

What we have done is to construct endomorphisms of the infinite lattice algebra ($N \to \infty$) defined by (17). Strictly speaking we have constructed maps from the $\mathcal{N} - 1$ generator algebra, say, to the $n\mathcal{N} - 1 (\approx 2N - 1)$ generator algebra, which become endomorphisms in the limit. The maps correspond to the existence of subgroups of the permutation group on $n\mathcal{N}$ objects (say), $S_{n\mathcal{N}}$, which do not permute within a cluster of n objects. More generally $S_{n\mathcal{N}} \supset S_{\mathcal{N}} \otimes (S_n \times S_n \times \ldots \times S_n)$, which we will denote $S_{\mathcal{N}} \otimes (S_n)^{\otimes \mathcal{N}}$. Strictly speaking the first product is semi-direct (the $S_{\mathcal{N}}$ permutes the S_n subgroups, so in this particular case we have the wreath product $S_n \mathscr{W} S_{\mathcal{N}}$). The distinction need not concern us here, and we will not make it in what follows. We are simply indicating how the $S_{\mathcal{N}}$ subgroup is realised. A different (non-unique) realisation is indicated, for example, by $S_{n\mathcal{N}} \supset S_{\mathcal{N}} \times S_{(n-1)\mathcal{N}}$. In terms of restriction of representations (Robinson 1961) we restrict the Potts representation R (say) for $S_{n\mathcal{N}}$:

$$\begin{aligned} RS_{n\mathcal{N}} \downarrow S_{\mathcal{N}} \times (S_1)^{\otimes (n-1)\mathcal{N}} &= \sum_{\mu} d_{\mu} \mu S_{n\mathcal{N}} \downarrow S_{\mathcal{N}} \\ &= \sum_{\mu} d_{\mu} \sum_{\nu} \hat{d}_{\mu\nu} \nu S_{\mathcal{N}} \end{aligned} \tag{22}$$

(the sums are over irreducible representations, νS_n) where d_{μ} is given in Martin (1988) and $\hat{d}_{\mu\nu}$ in this case by the usual skew tableau restriction rules. Not all realisations give the same $\hat{d}_{\mu\nu}$. In particular, the kind of representation responsible for the free energy in Potts and related models (tableaux of two equal rows) does not necessarily appear in the restriction to $S_{\mathcal{N}}$ of the corresponding representation of $S_{n\mathcal{N}}$, in the cabling realisation. For large n or $\mathcal{N}$, however, it typically does.

The determination of $\hat{d}_{\mu\nu}$ here is an interesting problem. One approach, suggested by R A Wilson, is via the conjugate but not identical realisation indicated by the sequence

$$S_{n\mathcal{N}} \supset S_{\mathcal{N}} \otimes S_{\mathcal{N}} \otimes S_{\mathcal{N}} \ldots \otimes S_{\mathcal{N}} \supset \text{diagonal } S_{\mathcal{N}}.$$

The procedure here is still rather complicated (because of the last restriction). However this conjugate alternative does serve to illustrate the point that the model will be invariant under many (if one) blockings, associated with conjugate realisations of the subgroup. Presumably not all these blockings correspond to a global scale change.

Note, however, that there is no semidirect product with $S_{\mathcal{N}}$ appropriate for irregular cablings, i.e. no composition law for elements of the *set* $S_{\mathcal{N}} \times (S_{n_1} \times S_{n_2} \times \ldots \times S_{n_{\mathcal{N}}}) \subset S_{n\mathcal{N}}$ where $\Sigma_{i=i}^{\mathcal{N}} n_i = n\mathcal{N}$. This is a pity, since such a group would allow the most obvious possibility of realising other than global conformal transformations explicitly. Note the possibility of an inverse dilatation interpretation of the representation induced from the subgroup to the group and related to the restricted representation above by Frobenius reciprocity (see Robinson 1961). We will leave these points for now, as they do not have an obvious parallel in the block spin picture.

6. Other cases

When $q \neq 4$ things are slightly more complicated. It is convenient to use the alternative set of generators $\{t_i\}$ for the Hecke algebra. Note that this presentation manifests the similarity with the braid group and is useful for working in this context, while the extra relation for the TL algebra

$$1 - t_i - t_{i+1} + \{t_i, t_{i+1}\} - t_i t_{i+1} t_i = 0 \tag{23}$$

is more simply expressed when working with the $\{U_i\}$ (equation (7*b*)).

We see that the map defined by (17), $\{t_i^{(n)}\}_m \to \{t_i\}_{nm}$, preserves the first two relations (12) (this map corresponds to cabled braid subgroups of the braid group: $\mathcal{B}_{nN} \supset \mathcal{B}_N \ltimes (\mathcal{B}_n \times \mathcal{B}_n \times \ldots \times \mathcal{B}_n)$) but not the third. In general the map will result in the replacement of (12*c*) and any other additional relations (such as (23)) with a different set of relations. Since $t_i^{(n)}$ is built from $2n-1$ consecutive generators from $\{t_i\}$, which generate at most (i.e. in the Hecke case) a $((2n)!)$-dimensional algebra, the new local relation (on $t_i^{(n)}$) will be polynomial of order at most $(2n)!$. In practice the order is much less than this, but we have not computed many specific cases.

Generically (see Martin 1989) the smallest faithful representation of a Hecke algebra is a direct sum of one copy of each inequivalent irreducible. The generic polynomial for $t_i^{(n)}$ can therefore be sensibly written out as a product of the factors coming from each irreducible representation of $\mathrm{H}_{2n-1}(q)$ (which generates $t_i^{(n)}$ up to isomorphism, by (12*d*)). This involves some degeneracy (multiple roots in the polynomial), but each such factor gives the polynomial local relation obtained from a corresponding non-trivial quotient of the original algebra. For $n = 2$ the dimensions of irreducibles are $1 \oplus 3 \oplus 2 \oplus 3 \oplus 1$ and the polynomial relation on $t = t_i^{(2)}$ is, by direct computation,

$$(t - Q^4)[(t + Q^3)(t + Q^2)][(t - Q)(t - Q^3)][(t - Q)(t + Q)(t + Q^2)](t - 1) = 0.$$

In this notation the $n = 1$ case is just $(t + Q)(t - 1) = 0$.

Besides (23), the kind of additional initial relations that appear include, in the Beraha cases ($q = 4\cos^2(\pi/r)$, $r = 3, 4, 5 \ldots$),

$$\text{idem}_i[r-1] = 0 \tag{24}$$

where $\text{idem}_i[r-1]$ is defined recursively by

$$\text{idem}_i[r-1]=\text{idem}_i[r-2]\left(1-\frac{\sinh((r-2)\theta)}{\sinh((r-1)\theta)}\,U_{i+r-3}\right)\text{idem}_i[r-2]$$

with $\text{idem}[1]=1$ (Martin 1988). It is easy to show that these relations restrict to unitarisable representations of the TL algebra. These are the additional relations which appear in the Potts model. The stronger the initial relations are, the stronger the relations will be at the renormalised level (this corresponds to the dependence on initial parameter values for a single transformation in the renormalisation group picture (Ravndal 1976)).

For example, we note that in general

$$t_i^k=\left(\sum_{j=0}^{k-1}(-e^{2\theta})^j\right)t_i-\left(\sum_{j=1}^{k-1}(-e^{2\theta})^j\right) \tag{25}$$

so that $t_i^k=1$ if $q=4\cos^2((k-2)\pi/2k)$, and in particular $t_i^4=1$ for $r=4$, $q=2$. If we impose the relations (24) in this case we find that, as far as we have checked ($n=1,\ldots,5$),

$$(t_i^{(n)})^4=1 \tag{26}$$

although the stronger relation (12*c*) is no longer satisfied for $n>1$ (see also the appendix). If we do not impose relations (24) then a higher-order polynomial relation is satisfied by $t_i^{(n)}$.

Thus, unless a sufficiently restrictive set of relations are preserved under the transformation (18), we have to check explicitly that, for a given representation of the $\{U_i\}_{nm}$, the constructed representation of the $\{t_i^{(n)}\}_m$ contains the representation of $\{t_i\}_m$ we want. The stronger the relations which survive after the transformation, the better our chances (see also part (ii) of the appendix).

We will give two examples. Firstly consider the Burau representation (Birman 1974) for the $\{t_i\}_{nm}$ satisfying (12*a*–*c*) and some other relations, defined by the *nm*-dimensional matrices:

$$({}_Bt_i(e^\theta))_{jk}=\begin{cases}1 & j=k\neq i\\ e^\theta & \\ -e^{2\theta} & j,k\begin{cases}=i,\,i+1\\ =i,\,i\\ =i,\,i-1\end{cases}\\ e^\theta & \\ 0 & \text{otherwise.}\end{cases} \tag{27}$$

Here we find that $t_i^{(n)}$ can be decomposed into irreducible components as follows:

$$t_i^{(n)}={}_Bt_i(e^{n\theta})\oplus\ldots. \tag{28}$$

In other words, this reducible representation of some larger algebra (larger in the sense that the Burau part of the TL algebra is a quotient) contains an irreducible component which gives the Burau representation of the TL algebra, again provided

$$e^\theta=e^{n\theta}$$

i.e.

$$q=4\cos^2(2\pi k/(n-1))\qquad (k\text{ any integer}).$$

Note again that, not only are we restricted to the Beraha q values, but that each of these only appears for certain values of n (the scale change factor). In fact the physical role of the Burau representation changes as the lattice size changes, so the significance of this n dependence is obscured.

Secondly, consider the Potts representation (Baxter 1982) for $q=2$. As we have said the $\{t_i\}$ obey relations (12), (23) and (24), which can be written

$$t_i^2 t_{i+1}^2 = -t_{i+1}^2 t_i^2. \tag{29}$$

Considering the case $\{t_i, i=1, \ldots, 2n\} \leftarrow \{t_i^{(n)}, i=1,2\}$ we recall that $\{t_i^{(n)}\}$ obeys (26), (12a) and (12b). In this case the quotient group of the three-string braid group is finite (96 elements) and we find that (for $n \geqslant 3$) the reducible representation of the $\{t_i^{(n)}\}$ and hence of this group constructed from the $\{t_i\}$ contains irreducible representations which satisfy the Potts relations (in addition to ones which do not). There is evidence to suggest that this generalises to $\{t_i^{(n)}, i=1, \ldots, m\}$, but the calculations are presently rather tricky.

The significance of such representations is that they imply that parts of the spectrum of the transfer matrix for the corresponding critical statistical mechanical model (5) are invariant under the appropriate dilatations in the thermodynamic limit (equivalent representations give identical contributions to the spectrum).

A natural question to ask at this stage is: why not linearise about the fixed point and obtain the critical exponents? Unfortunately, as we can see from (13a), different parts of $\tilde{U}_i$ transform in different ways under $e^\theta \to e^\theta + \varepsilon$. Thus there is no unambiguous response to measure. We have to know how to rewrite $\tilde{U}_i$ away from criticality, which, as yet, we do not. There are various procedures we could follow, but none are compelling. In any case we already *know* the exponents from the conformal programme, what we have done is to begin to show why this programme is relevant!

The next step in this process is the systematic computation of the evolution of the quadratic term (12c) under $\{t_i^{(n)}\} \to \{t_i\}$. This is achieved by constructing a faithful representation of $t_i^{(n)}$ using the known representations of $\{t_i\}$ (see Martin 1988, for example), and finding the characteristic polynomial. By checking if the polynomial contains (12c) as a factor we can immediately see if the weak version of (4) may be satisfied. This is a tractable procedure in principal, but a substantial work of computer algebra (even for the small cases) in practice.

Acknowledgments

I would like to thank Rob Wilson and Bruce Westbury for useful discussions.

Appendix

(i) Note that the transfer matrix may be written in terms of the $\{t_i\}$ at other temperatures. We proceed formally by

$$t_i^k = (1 - e^\theta U_i)^k = 1 + f(e^\theta, k) U_i \qquad k \in \mathbb{Z}$$

but f has a natural extension from $k \in \mathbb{Z}$ to $k \in \mathbb{C}$ so that for $(1 + xU_i)$ we have $k = f^{-1}(e^\theta, x)$. We use the relation (11) to recover an element of the algebra from such objects when k is not integer.

(ii) It is interesting to consider the operator algebraic structure behind the partial dilatation invariances we have found. It is easy to see that $(t_i^{(n)})^2$ commutes with t_j for $j=ni-n+1$ to $ni+n-1$ (excepting $j=ni$) because, with $\mathscr{B}_{nN} \supset \mathscr{B}_N \times (\mathscr{B}_n)^{\otimes N}$, the object $(t_i^{(n)})^2$ has no action in the $(\mathscr{B}_n)^{\otimes N}$ (it is an element of the pure braid subgroup of the $\mathscr{B}_N$ and the $\mathscr{B}_{nN}$ (Birman 1974)) while these t_j act purely within a $\mathscr{B}_n$. Furthermore

$$C_n = (t_1 t_2 \ldots t_n)^{n+1}$$

is central in $\mathscr{B}_{n+1}$, i.e. it commutes with all the $\{t_i, i=1, n\}$ (conjugation by C_n corresponds to translation through $n+1$ steps in an $n+1$ site periodic lattice, since $C_n = (T')^{n+1}$; note that T' thus has all eigenvalues of constant magnitude in any irreducible representation). Now, if $n = 2^j$ then

$$\chi_{ni} = t_i^{(n)}(t_{2i+1}^{(2^{j-1})} t_{2i-1}^{(2^{j-1})})(t_{4i+3}^{(2^{j-2})} t_{4i+1}^{(2^{j-2})} t_{4i-1}^{(2^{j-2})} t_{4i-3}^{(2^{j-2})}) \ldots (t_{ni+n-1} \ldots t_{ni-n+1})$$

has the property that every factor $t_a^{(b)}$ braids within a cable of some $t_c^{(2b)}$ and $(\chi_{ni})^2 = C_{2n-1,i}$ where $C_{2n-1,i}$ is C_{2n-1} with t_k replaced by t_{k+ni-n}. This means, for instance, that for $q=2$

$$(t_1^{(2)})^4 = (C_3)^2$$

and so on (we have used $t_i^4 = 1$). In other words, although $(t^{(2)})^4$ is not necessarily unity it is a central element in the algebra of three operators. Similarly (using (24))

$$(t^{(3)})^4 = (C_5)^2.$$

These observations are peculiar to $q=2$. Other Beraha q values have analogous relationships. This exercise inaugurates a search for substructure in $H_{n-1}(q)$ (q Beraha) analogous to the cabled braid subgroups of $\mathscr{B}_n$, which is realised for $q=4$ in the permutation group subgroups. This is an exciting programme, both physically and mathematically.

(iii) There are intriguing subalgebras of the TL algebra obtained by replacing t_i with U_i everywhere in (17), whereupon the new $\{U_i^{(n)}\}$ (distinct from the $U_i^{(n)}$ obtained via (17) as it stands) obey relations (7) with $q \to q^n$. Note that, for $q=0, 1$, these dilatation-like transformations again map the infinite lattice algebra to itself. There is no obvious connection with the cabled braid subgroup picture, but for the Potts models this looks like a block spin transformation in which the new (q^n state) variables simply encode the configuration of n q-state variables.

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